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WALL INTERFERENCE IN A PERFORATED WIND TUNNEL

By Riccardo Brescia

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SUMMARY

The theory of cascades, made up of a series of flat plates placed one behind the other, is extended to the case where the impinging stream is not uniform, and the deduced properties of this cascade-flow are then applied to the study of the wall interference between such a cascade-like boundary and a vortex-source type of singularity. It is shown that the induced velocities, produced by the presence of such a wall, are equal to what is obtained by action of a suitably chosen "reflected" singularity situated on one side of the wall, together with the action of another suitably chosen "transmitted" singularity placed on the other side.

The concepts of a reflection factor and a transmission factor are introduced to characterize various types of tunnel boundary, whether this is composed of solid wall, open-and-closed sequences, or just a free fluid surface. These ideas are then extended to cover the situation in which a pair of such walls are allowed to coalesce, expecially in the event that one of the walls consists of open-and-closed portions and the other is a fluid surface. This latter particular combination of boundaries is called a perforated wall.

Finally, the interference arising from an arbitrary general singularity placed symmetrically in between two such perforated walls is analyzed, and these results are then applied to the determination of the particular kinds of wall geometry that will produce no interference effects in the case of a slender lifting wing and also in the case of a symmetric profile having a finite thickness.

CHAPTER I

1.   Let attention be directed to a straight wall made up of a series
of like-sized flat plates, each of chordlength L, and following one
right after another, with a distance between centers of D. (See
fig. 1.)

Let y be the perpendicular distance of any arbitrary general
point from the straight wall. Furthermore, take $Y = y/D$ as the ratio
of the normal distance to the "pitch" (or interval between the succes-
sive repetitions of the gapped wall).

The behavior of such a cascade of plates at an infinite distance
above and below the wall (i.e., at the locations where $Y = \pm\infty$), that
is under action of an impinging stream, with a uniform undisturbed
velocity, denoted by the vector $\vec{V}_{1\infty}$, is described by means of the two
equations

$$\left.\begin{aligned}
\vec{V}_{m\infty} &= \vec{V}_{1\infty} - \vec{V}_{s\infty} \\
\vec{V}_{v\infty} &= \vec{V}_{1\infty} + \vec{V}_{s\infty}
\end{aligned}\right\} \tag{1.1}$$

where the velocity below the wall is denoted by the m subscript and the
velocity in the upper region by the v subscript. Of course, $\vec{V}_{s\infty}$
stands for the induced velocity, produced by action of the gapped wall,
in a direction parallel with the length of the wall. The magnitude of
this induced velocity is provided by cascade theory and has the value

$$V_{s\infty} = qV_{n\infty} \tag{1.2}$$

where $V_{n\infty}$ is the component of $\vec{V}_{1\infty}$ which is <u>normal</u> to the plates,
while q is related to the L/D ratio through the relationship

$$q = \tan\left(\frac{\pi}{2}\frac{L}{D}\right) \tag{1.3}$$

Provided it is understood that the $q\vec{V}_{n\infty}$ vector is to be taken
as lying in the direction <u>parallel</u> with the horizontal extent of the
cascade-like wall, one may thus rewrite equations (1.1) as

$$\left.\begin{aligned}
\vec{V}_{m\infty} &= \vec{V}_{1\infty} - q\vec{V}_{n\infty} \\
\vec{V}_{v\infty} &= \vec{V}_{1\infty} + q\vec{V}_{n\infty}
\end{aligned}\right\} \tag{1.4}$$

The sole condition which it is necessary to stipulate in order that
the equations (1.4) should hold true is that $Y = y/D$ must be infinitely
large. It may be observed that these equations also remain unaltered
provided that the L/D ratio stays constant. In view of these facts,
one may proceed to treat the case where y is merely of finite size,
however, by simply letting L and D tend toward zero in magnitude,
but in such a manner that the ratio between them is retained at a fixed
constant value. The assumption will henceforth be explicitly adopted
that the pitch and the chordlength are, in fact, now both to be infini-
tesimally small, in comparison with any other of the distances that
might enter into the argument. In consequence of this assumption, there-
fore, it follows that equations (1.4) become converted into just

$$\left.\begin{array}{l} \vec{V}_m = \vec{V}_1 - q\vec{V}_n \\[2mm] \vec{V}_v = \vec{V}_1 + q\vec{V}_n \end{array}\right\} \tag{1.5}$$

and the gapped wall should thus be looked upon as merely a plate which
is refracting the streamlines.

On this basis it may also be assumed that the relationships given
as equations (1.5) will not only hold true for a uniform stream but that
they will, in addition, also be valid for any velocity distribution
whatsoever, provided merely that the pitch and chordlength are taken
to remain infinitesimally small with respect to the radii of curvature
of the streamlines existing in the flow just below and just above the
perforated wall.

2. Now let the flow field be referenced to a complex planar coordinate
system such that the (reals) x-axis coincides with the direction of the
straight wall, while the (imaginaries) iy-axis is in the direction lying
perpendicular to the wall; the origin is taken to be at any arbitrary
location. (See fig. 1.) The "outside" region of the flow is thus the
infinite half-plane for which $y > 0$, while the "inside" portion of the
flow is the infinite half-plane corresponding to the points for which
$y < 0$. Then upon rewriting the vector operations given as equations (1.5)
as separate scalar equations, it is seen that

$$\left.\begin{array}{ll} V_{mx} = V_{1x} - qV_{1y} \quad \text{and} \quad V_{vx} = V_{1x} + qV_{1y} \\[2mm] V_{my} = V_{1y} \qquad\qquad\qquad V_{vy} = V_{1y} \end{array}\right\} \tag{2.1}$$

and from comparison of the forms of these relationships it follows that

$$V_{my} = V_{vy} = V_y \tag{2.2}$$

and

$$V_{vx} - V_{mx} = 2qV_y \tag{2.3}$$

The behavior of the flow in passing through the cascade-like wall is thus completely defined by means of equations (2.2) and (2.3).

The fact that the presence of the cascaded wall does not change the normal component of velocity at all is rather surprising. On the other hand, it is apparent that the tangential component of the flow undergoes a sudden jump as the flow passes the grid pattern of the wall. The magnitude of this jump in the tangential velocity component is proportional to the normal component of the impinging stream. Now this change in the tangential velocity-component, brought about by the gapped wall, may be interpreted as though it were actually due to the action of a distribution of vortices placed along the x-axis and having a strength γ, whose magnitude is given by

$$\gamma = V_{mx} - V_{vx} = -2qV_y \tag{2.4}$$

For the sake of convenience, let a running coordinate along the x-axis be denoted by the symbol ξ. Then the expression for the complex potential governing the distribution of velocities generated by the presence of the gapped wall is readily seen to be

$$w = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} 2qV_y(\xi) \, \log \, (z - \xi) \, d\xi \tag{2.5}$$

It should be pointed out that everything that has been said so far is valid only if the flow field is isoenergetic, and it will be taken for granted in what follows that this **is** the situation under examination, so long as the case of the mixed boundary condition (the perforated tunnel) is not being treated.

In what follows, it will be very convenient to make use of the expression for the complex velocity, which may be written down at once as

$$\frac{dw}{dz} = -\frac{q}{\pi i} \int_{-\infty}^{\infty} V_y(\xi) \, \frac{d\xi}{z - \xi} \tag{2.6}$$

All possible aerodynamic configurations which are met in practice can be built up out of the fundamental situation wherein there exists in the field of flow one single arbitrary general singularity of the

form

$$A = Q + \frac{1}{i} \Gamma \qquad (2.7)$$

(i.e., a combination of a source of strength Q with a vortex of intensity Γ). In what follows it will be sufficient, therefore, to concentrate attention on the study of the flow field resulting from the presence of such a singularity, A, placed at the position denoted by $-iY_0$ (see fig. 1) from the cascade-like wall, and immersed in a stream flowing parallel with the x-axis, the magnitude of whose velocity vector is taken to be V_∞ at infinity.

The complex velocity corresponding to such a flow field is seen to be (with the aid of eq. (2.6)):

$$\frac{dW'}{dz} = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_0} - \frac{q}{i\pi} \int_{-\infty}^{\infty} V_y(\xi) \frac{d\xi}{z - \xi} \qquad (2.8)$$

in which it is plainly seen that

$$V_y(\xi) = -\underline{I} \left(\frac{dW'}{dz} \right)_{z=\xi} \qquad (2.9)$$

holds true.

3.   Consequently, it is clear that equation (2.8) turns out to be just an integral equation in the complex velocity. The solution of this equation will be attacked, however, by making use of an indirect route, wherein one relies on the analogy which exists between this case and the similar problem with which one is confronted when dealing with a solid wall for the boundary (thus the solid-wall example represents the limiting case of cascade-type boundary, for which the chordlength of each portion of the wall becomes equal to the "pitch", that is to say, in the limiting case, one has $L/D = 1$, and $q = \infty$).

In this analogous solid-boundary case, it is well known that the requisite distribution of vortices lying along the x-axis that gives the proper solution to the problem inside the tunnel $(y < 0)$ is equivalent in all respects to a singularity $\bar{A}$ (conjugate of A) placed at the point $+iY_0$ (which is the image point of $-iY_0$ with respect to the straight wall), while the requisite distribution for those points lying outside $(y > 0)$ is equivalent to a singularity $-A$ placed at the point $-iY_0$ (which thus cancels the effect of the original singularity located there).

Now this equivalence, which is evinced in this special case where one is dealing with a solid boundary, suggests the possibility immediately of extracting therefrom a generalization which will be applicable even in the case where one is concerned with the gapped-wall type of boundary. In fact the natural thing to do is to take the expression for the complex velocity corresponding to a field point lying "inside" in the form

$$\frac{dW_m{}''}{dz} = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_0} + \frac{A'}{2\pi} \frac{1}{z - iY_0} \qquad (3.1)$$

wherein A' stands for a suitable singularity located at the point $+iY_0$ (the image-point of $-iY_0$). The value of this latter singularity may be determined by invoking the condition that equations (2.8) and (3.1) should be identical; that is to say, it must be true that

$$\frac{dW_m{}''}{dz} = \frac{dW_m{}'}{dz} = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_0} - \frac{q}{\pi i} \int_{-\infty}^{\infty} V_y(\xi) \frac{d\xi}{z - \xi} \qquad (3.2)$$

But it can be seen by reference to equation (3.1) that the value of $V_y(\xi)$ is linked to the singularities from equation (2.9) by the relation:

$$V_y(\xi) = \frac{1}{2i} \left[\left(\overline{\frac{dW_m{}''}{dz}} \right)_{z=\xi} - \left(\frac{dW_m{}''}{dz} \right)_{z=\xi} \right]$$

$$= \frac{1}{4\pi i} \left[\frac{\overline{A}}{\xi - iY_0} + \frac{\overline{A}'}{\xi + iY_0} - \frac{A}{\xi + iY_0} - \frac{A'}{\xi - iY_0} \right] \qquad (3.3)$$

This value of $V_y(\xi)$ is now to be inserted into the integral expression arising in equation (3.2). The indicated integration is actually carried out merely by integrating along the whole x-axis, entirely within the domain of reals. In order to make use of the powerful methods of the complex variable, it should be observed, however, that the integrand is always going to be an infinitesimal of the second order along the whole circle at infinity in the complex plane. Then, if it is formally agreed to perform the integration in more generality around a closed contour in the complex plane, consisting of the x-axis and the semicircle at infinity for values of $y > 0$, and if the direction of the path of the integration is taken to be that of the positive sense for the variable of integration, ξ (which is now considered to be complex), it follows that the sought value of the real integral is equivalent to an integration in the

complex plane, which can be carried out by having recourse to the theorem of residues, to give

$$
\int_{-\infty}^{\infty} V_y(\xi)\ \frac{d\xi}{z-\xi} = \frac{1}{4i\pi} \oint \left[\frac{\overline{A}}{\xi - iY_0} + \frac{\overline{A}'}{\xi + iY_0} - \frac{A}{\xi + iY_0} - \frac{A'}{\xi - iY_0} \right] \frac{d\xi}{z-\xi}
$$

$$
= \frac{1}{2} \frac{\overline{A}}{z - iY_0} - \frac{1}{2} \frac{A'}{z - iY_0} \qquad\qquad (3.4)
$$

Thus, the complex velocity of the flow now under examination may be written down at once, by referring to equation (3.2), as

$$
\frac{dW_m'}{dz} = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_0} + \frac{q}{2\pi i} \frac{1}{z - iY_0} (A - \overline{A}') = \frac{dW_m''}{dz} \qquad (3.5)
$$

Hence, because of the equality which must exist between the complex velocities for the two flows, one is led to the conclusion that

$$
A' = \frac{q}{i} (A' - \overline{A}) = \overline{A} \frac{q}{q - i} \qquad\qquad (3.6)
$$

It is obvious, therefore, that the reflected singularity A', when having the value just determined, will satisfy the requirements imposed on the behavior of the flow in consequence of the presence of the gapped wall. Hence, the problem which was set has been formally solved in closed form. For the sake of convenience, in what follows, the above result will be written more compactly as

$$
A' = r\overline{A} \qquad\qquad (3.7)
$$

where r is defined as the reflection factor of the wall. Thus it is seen that

$$
r = \frac{q}{q - i} = \frac{q^2}{q^2 + 1} - \frac{1}{i} \frac{q}{q^2 + 1} \qquad\qquad (3.8)
$$

This result may be interpreted by means of the statement: The "reflected" singularity for a gapped wall is equivalent to the reflection factor (complex) multiplied by the complex conjugate of the original singularity.

If one now goes on to investigate in like manner what happens when the field point lies outside of the wall, where $y > 0$, it is necessary to carry out the integration indicated in equation (3.4), in the complex

plane, by going around the closed curve composed of the entire x-axis
and the semi-circle at infinity lying in the region where $y < 0$. The
sense of this integration is opposite to that of ξ in this instance,
and the z belongs now to that half of the plane where $y > 0$. Thus
the value of the integral becomes now

$$\int_{-\infty}^{\infty} V_y(\xi) \, \frac{d\xi}{z - \xi} = -\frac{1}{4\pi i} \oint \left[\frac{\bar{A}}{\xi - iY_O} + \frac{\bar{A}'}{\xi + iY_O} - \frac{A}{\xi + iY_O} - \frac{A'}{\xi - iY_O} \right] \frac{d\xi}{z - \xi}$$

$$= \frac{1}{2} \frac{A}{z + iY_O} - \frac{1}{2} \frac{\bar{A}'}{z + iY_O} = \frac{1}{2} \frac{A}{z + iY_O} \left(1 - \frac{q}{q + 1} \right) \quad (3.9)$$

The complex velocity pertaining to the downstream flow is thus
obtained by inserting the above expression into equation (2.8), with
the result that

$$\frac{dW_v'}{dz} = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_O} - \frac{A}{2\pi} \frac{1}{z + iY_O} \frac{q}{i} \left(1 - \frac{q}{q + 1} \right)$$

$$= V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_O} \left(1 - \frac{q}{q + 1} \right) = V_\infty + \frac{A}{2\pi} \frac{1}{z + iY_O} (1 - \bar{r})$$

$$(3.10)$$

That is to say, viewed from a point outside the gapped wall, the
field of flow can be looked upon as though it were generated by a single
singularity $A(1 - \bar{r})$ located at the point $-iY_O$. In analogy to the
practice common in optics, this will be called a "transmitted" singu-
larity. The magnitude of this singularity may be interpreted as being
the product of the magnitude of the original singularity and the trans-
mission factor $\tau = 1 - \bar{r}$. It then may be noted that

$$\tau = 1 - \bar{r} = 1 - \frac{q}{q + 1} = \frac{1}{q^2 + 1} + \frac{1}{i} \frac{q}{q^2 + 1} \quad (3.11)$$

It is worth especial notice to point out the extraordinarily remark-
able fact that each reflection leads once again to use of the complex
conjugate of the original singularity, but nothing of a similar nature
arises when one works with the transmitted singularity.

Making use of this observation, one may now easily see the inter-
connections in the singularities which one obtains when the original
singularity lies either above or below the gapped wall. To do this,
let a sketch be prepared that shows the location of the singularities

and the trace of the gapped wall, lying parallel with the x-axis. (See fig. 2.) Now imagine that this scheme of singularity-placement is reflected in a line lying parallel with the x-axis. This reflection is made in such a way that each quantity will have a complex conjugate, as indicated.

For simplicity's sake the singularities in the flow which have thus far been under examination are written with a bar over them. This is done in order to obtain a result which will be directly valid for the reflected image of the field of flow just about to be studied, wherein, for sake of preciseness, it will be assumed that the original singularity is located at a point lying above the gapped wall. If the original singularity lies beneath the gapped wall, it can be seen, by reference to the relationships already sketched in figure 2, that the only thing which needs to be modified is that the reflection and tranmission factors should be taken to be the complex conjugates of the corresponding reflection and transmission factors applying in the case where the gapped wall is located below the original singularity.

4. Everything that has so far been derived may be summarized quite succintly merely by the statement: The presence of a gapped wall in the field of flow coming from the singularity A is equivalent to: if the field point lies inside, a "reflected" singularity $r\bar{A}$ located at the image point, while if the field point lies outside, then a flow is generated which is equivalent to the presence of a "transmitted" singularity τA located at the place where the original singularity was situated.

The factors of reflection and of transmission can thus be written as (provided one makes use of a double-level sign, of which the upper is to be selected if the singularity lies above the gapped wall, and the lower of which applies if the singularity lies below the gapped wall):

$$r = \frac{q}{q \pm i} = \frac{q^2}{q^2 + 1} \pm \frac{1}{i}\frac{q}{1 + q^2} \qquad (4.1)$$

$$\tau = 1 - \frac{q}{q \mp i} = \frac{1}{q^2 + 1} \pm \frac{1}{i}\frac{q}{1 + q^2} \qquad (4.2)$$

These factors are linked to each other by the relations

$$\tau = 1 - \bar{r} \qquad (4.3)$$

$$r = 1 - \bar{\tau} \qquad (4.4)$$

Of course, as was said earlier in Article 1, the value of q is obtained from

$$q = \tan \left(\frac{\pi}{2} \frac{L}{D} \right) \tag{4.5}$$

where L stands for the solid chord length of each open-and-closed piece of the gapped wall, and where D represents the "pitch", or interval between sequential pieces of the breached wall.

Since all possible singularities, either of a concentrated or distributed type, can be considered as acting like sources and vortices, the results just obtained can be made applicable to all aerodynamic singularities.

As a sort of check, it can be noted, through recourse to equation (4.1), that when L/D approaches unity (the solid-wall case) then q becomes infinite and r becomes unity too; that is to say, when L/D = 1, the "reflected" singularity is simply equal to Ā, regardless of what the relative location of the singularity is with respect to the wall. On the other hand, the "transmitted" singularity in this case is τA, where τ = 0, and thus this result jibes with the situation with which one already is familiar.

If the L/D ratio tends toward zero, however, which means that the wall effectively disappears (but it does not become a fluid surface, because it is being assumed that the flow is isoenergetic), then q becomes zero, and consequently r = 0 and τ = 1. In this case, therefore, there is no reflection taking place and the "transmitted" singularity is τA = A, and it thus coincides with the original one. This is quite obviously the correct result, because if nothing is present to disturb the flow, it remains the same as it started. Hence, the above derived theory appears to be fully justified, and accords with known results in the two limiting cases thus tested.

5.   It was explicitly made clear in the above treatment that only the behavior of those interactions was being examined which apply strictly to the case of a gapped wall placed in isoenergetic flow. In actuality, however, a perforated wind tunnel would require the study of nonisoenergetic fields. This is so because the real perforated tunnel can be thought of as being a superposition of a free fluid surface (that is, this boundary is just the line of demarcation between the tunnel jet and the external atmosphere) upon the gapped-wall type of flow. When two such boundaries are allowed to coalesce, they constitute together what will now be called a perforated-wall tunnel, or simply a perforated wind-tunnel.

It is well known that the effect of a free fluid surface (two-dimensional) operating in conjunction with a singularity A is equivalent to the presence of a singularity $-A$ placed at the image point of A; that is, the reflection factor for a free fluid surface is

$$r' = -1 \qquad (5.1)$$

and consequently

$$\tau' = 1 - \overline{r}' = 2 \qquad (5.2)$$

In order to study most conveniently the case of such mixed boundaries, it is best at first merely to handle each wall separately and then let them coalesce. By aid of this process of bringing the two walls into juxtaposition, one may thus deduce what the appropriate reflection factor should be for such "mixed" walls.

To proceed with this analysis of the double walled tunnel, it is convenient to denote the gapped wall as P_s and the free fluid surface as P_f. Also let $A = Q + \Gamma/i$ represent a singularity located inside the wall. (See fig. 3.)

It should be recalled that the selfsame wall will have reflection and transmission factors of r and τ, respectively, or of $\overline{r}$ and $\overline{\tau}$ depending upon whether the radius vector, which originates at the singularity which is giving the reaction at the wall, approaches the wall in question from above or from below.

If one imagines that the radius vector emanating from the singularity A has the same attributes as would appertain to a ray of light issuing from this source, it is quite easy to find the locations of the multiple reflections R_s, R_{f_1}, R_{f_2}, . . . which are produced by action of the two walls under study, and at the same time it will be equally easy to establish where the points of incidence are located depending upon whether the ray approaches them from below or above.

By use of a system of notation which is perfectly obvious, it may be stated that the first reflection point is thus given by the relation

$$R_s = \overline{r}_s \overline{A} \qquad (5.3)$$

In order to arrive at the point f_1, the ray u must undergo a process consisting of a transmission at P_s characterized by the factor $\overline{\tau}_s$; a reflection at P_f characterized by a reflection factor $\overline{r}_f$; and in turn another transmission at P_s characterized by the factor τ_s. As a result, the ray in question now appears to issue from the reflection

point R_{f_1}, and the strength of this reflected singularity is thus given as

$$R_{f_1} = \overline{(A\bar{\tau}_s)}\bar{r}_f\tau_s = \overline{Ar_f}\tau_s{}^2 \qquad {}^1 \qquad (5.4)$$

In order to arrive at the point f_2, the ray must undergo a process consisting of three reflections and two transmissions. As a result the ray will then appear to issue from the reflection point R_{f_2}, and the strength of this singularity will be found, according to a process which is entirely analogous to what was done in the previous case, to be

$$R_{f_2} = \left\{ \overline{\left[\overline{(A\bar{\tau}_s)}\bar{r}_f\right] r_s} \right\} \tau_s\bar{r}_f = \overline{Ar_f}\tau_s{}^2 r_f\bar{r}_s = R_{f_1}r_f\bar{r}_s \qquad (5.5)$$

The simple repetition of this process of tracing out the paths of the light ray thus leads one to write in general:

$$R_{f_{(n+1)}} = R_{f_n}r_f\bar{r}_s = R_{f_1}(r_f\bar{r}_s)^n = \overline{Ar_f}\tau_s{}^2(r_f\bar{r}_s)^n \qquad (5.6)$$

It is now time to see what happens when the two walls P_f and P_s are allowed to come closer and closer until they finally coincide. It so happens that now all the "reflected" singularities once again come together and coalesce in the image point of A, while their magnitudes are all gathered into one single "reflected" singularity A^*, the magnitude of which is

$$A^* = \bar{A}\left[\bar{r}_s + \sum_{n=0}^{\infty} r_f\tau_s{}^2(r_f\bar{r}_s)^n\right] = \overline{Ar_m} \qquad (5.7)$$

where $\bar{r}_m$ stands for the reflection factor of the perforated (or mixed) wall. By carrying out the indicated sum of the geometric series, it follows that (provided it is taken for granted that $\tau_s = 1 - \bar{r}_s$):

$$\bar{r}_m = \bar{r}_s + \bar{r}_f \frac{(1 - \bar{r}_s)^2}{1 - r_f\bar{r}_s} \qquad (5.8)$$

---

[1]The bar placed over the symbols in parentheses is used to indicate that one should take the complex conjugate of the indicated quantity.

In the case in which interest is now focussed, the wall P_f is to be taken as a free fluid surface, and thus the reflection factor which pertains to a perforated tunnel is obtained by making use of equation (5.8) (which is true in general) by particularizing it to the case wherein r_f is set equal to $\bar{r}_f$, and where $\bar{r}_f = -1$, and where, besides, $\bar{r}_s = \dfrac{q}{q-1}$. Thus now

$$\bar{r}_m = \bar{r}_s - \frac{(1 - \bar{r}_s)^2}{1 + \bar{r}_s} = \frac{3\bar{r}_s - 1}{\bar{r}_s + 1} = \frac{2q^2 - \frac{3q}{1} - 1}{2q^2 - \frac{1}{1}q + 1} \qquad (5.9)$$

It is also clear that if the perforated tunnel wall happens to be below the singularity, the reflection factor is the conjugate of the one just now deduced.

One may check the above-derived result by letting the L/D ratio vary from one limit value to the other and noting what transpires. For instance, let L/D = 1 first, and then $q = \infty$ (a solid wall and a free surface coming into coincidence) and thus the reflection factor in this case is $r_m = 1$. On the other hand, if L/D is allowed to be zero, then $q = 0$ (the solid wall disappears but a single free fluid surface remains), and in this case the reflection factor is $r_m = -1$. These results are just what one would expect.

It is worthwhile pointing out that $\left| r_m \right| = 1$, and therefore one may write

$$r_m = e^{i\vartheta} \qquad (5.10)$$

where

$$\vartheta = \text{argument } (r_p) = G(L/D) \qquad (5.11)$$

The value of this functional relationship $\vartheta = G(L/D)$ has been computed for a number of values of L/D, and the results are reported in the appended table for the case where the tunnel wall lies underneath the singularity A.

L/D	ϑ	L/D	ϑ
0	-3.14	0.5	-0.93
.05	-3.125	.6	-.722
.1	-2.54	.7	-.500
.2	-1.995	.8	-.324
.3	-1.55	.9	-.158
.4	-1.21	1.0	.00

CHAPTER II

1. In the next step towards understanding the action of a perforated
wind-tunnel, it is necessary to take into consideration the effect of
both of the perforated walls P and Q, one of which lies above and the
other lies below the singularity A. The distance between these walls
is assumed to be the height h, and they are symmetrically placed with
respect to the singularity, which is itself considered to lie at the
origin of the complex coordinate system to which the flow is referred.

It is clear that the usual multiple reflections also take place
here, and in consequence it will be true that the velocity at each point
in the region of the flow comprised between the two walls will be composed
of the sum of the undisturbed velocity V_∞ (the velocity vectors produced
by direct action of the singularity A, together with the velocities
arising from interference from all the reflected singularities).

Now let r_p and r_q be the reflection factors for the lower wall
and upper wall, respectively. Then if one examines the ray u_p emanating
from A and progressing downward from A (this path is denoted in fig. 4
by means of a solid line) it will be seen to be reflected at the wall P,
and because of this it will appear to have issued from the singularity
P_1 located at a vertical distance downward of -ih. The strength of
this singularity is given by the relation

$$P_1 = \overline{A}r_p \qquad\qquad (1.1)$$

Continuing on with the tracing out of the path followed by the
ray u_p, it will be recognized that it is again reflected from wall Q,
and it will thus appear to have come from the singularity P_1', located
at a vertical height of 2ih, and having a strength given by

$$P_1' = \overline{P}_1 r_q = \left(\overline{\overline{A}r_p}\right)r_q = A\overline{r}_p r_q \qquad\qquad (1.2)$$

It is intuitively obvious that the path of the ray u_p will be
periodically repeated in such a way that any two successive "reflected"
singularities P_n' and P_{n+1}' will have the same relation between each
other as the singularities P_1' and A have. Thus it will be true in
general that

$$P_{n+1}' = P_n'\overline{r}_p r_q \qquad\qquad (1.3)$$

and consequently it will be true that

$$P_n' = A\left(\overline{r}_p r_q\right)^n \tag{1.4}$$

while at the same time the ordinate location of the singularity P_n' will be given as

$$iy_{P_n}' = i2nh \tag{1.5}$$

Returning now once again to take up the question of the continuation of the path of the ray u_p after it has undergone the last reflection considered previously, it will be seen that it is once again reflected at the wall P. In consequence of this reflection the ray now will appear to be emanating from the singularity P_2 which lies at a distance $-i3h$ below the wall, and the strength of which is given by

$$P_2 = \overline{P_1'r}_p = \overline{A}r_p\overline{r}_q r_p = P_1\overline{r}_q r_p \tag{1.6}$$

It will again be abundantly clear, therefore, that any two successive "reflected" singularities P_{n+1} and P_n will be interrelated in exactly the same way that the singularities P_2 and P_1 were found to be connected above. Thus, in this case, it is obvious that

$$P_{n+1} = P_n\overline{r}_q r_p \tag{1.7}$$

and consequently it will be true that

$$P_n = P_1\left(\overline{r}_q r_p\right)^{n-1} = \overline{A}r_p\left(\overline{r}_q r_p\right)^{n-1} \tag{1.8}$$

and the ordinate distance of P_n is given as

$$iy_{P_n} = -i(2n - 1)h \tag{1.9}$$

By means of this procedure half of the reflections of the singularity at A have now been determined. In order to find the other half one may

proceed to follow an entirely analogous development which stems from the tracing out of the path followed by a ray u_q. The route taken by this ray will be entirely symmetric to the one traversed by u_p (this new path is also drawn in on fig. 4) except that now the order in which the walls are encountered by the ray u_q will be inverted from what was true for the u_p ray. It is quite evidently sufficient, therefore, merely to interchange the symbols r_p and r_q in the above-written formulas in order to arrive at the correct expressions for the magnitudes of the "reflected" singularities Q_n and Q_n', which lie at locations which are symmetrically placed with respect to the analogous singularities P_n and P_n'. The sought values appertaining to this half of the "reflected" singularities are thus obtained as:

$$Q_n = \overline{A}r_q\left(\overline{r}_p r_q\right)^{n-1} \tag{1.10}$$

$$iy_{Q_n} = i(2n - 1)h \tag{1.11}$$

$$Q_n' = A\left(\overline{r}_q r_p\right)^{n} \tag{1.12}$$

and

$$iy_{Q_n}' = -i2nh \tag{1.13}$$

2. It is now convenient to introduce the factor $f(z-z_0)$, which is the amount by which the magnitude of an arbitrary singularity located at the point z_0 must be multiplied in order to determine the magnitude of the complex velocity produced at the point z. By making use of this convention and by having recourse to the expressions just derived in the previous paragraph, one may write down the value of the complex velocity as a sum of all those complex velocities which are produced by action of the multiple reflections in the two perforated walls. This total value for the complex velocity is thus

$$\frac{dw}{dz} = \overline{A}\left\{\sum_{n=1}^{\infty} r_p\left(\overline{r}_q r_p\right)^{n-1}f\left[z + i(2n - 1)h\right] + \right.$$

$$\left.\sum_{n=1}^{\infty} r_q\left(\overline{r}_p r_q\right)^{n-1}f\left[z - i(2n - 1)h\right]\right\} +$$

$$A\left\{\sum_{n=1}^{\infty}\left(\overline{r}_p r_q\right)^{n}f(z - i2nh) + \sum_{n=1}^{\infty}\left(\overline{r}_q r_p\right)^{n}f(z + i2nh)\right\} \tag{2.1}$$

Since in actual practice it will be true that the most commonly met
tunnel configuration will be one for which both of the perforated walls
have the same reflection properties, and since there will be a good deal
of simplification resulting from such an assumption, it will be assumed
that this is the type of tunnel with which the following discussion will
be concerned. In this case, it is true that

$$r_p = \bar{r}_q \qquad\qquad (2.2)$$

and consequently, equation (2.1) will become simplified to just

$$\frac{dw}{dz} = \bar{A}\left\{\sum_{n=1}^{\infty} r_p^{2n-1} f\left[z + i(2n - 1)h\right] + \right.$$

$$\left. \sum_{n=1}^{\infty} (\bar{r}_p)^{2n-1} f\left[z - i(2n - 1)h\right]\right\} +$$

$$A\left\{\sum_{n=1}^{\infty} r_p^{2n} f(z + i2nh) + \sum_{n=1}^{\infty} (\bar{r}_p)^{2n} f(z - i2nh)\right\} \qquad (2.3)$$

One can bring about another great simplification in the work if
the complex velocity is only sought for points lying along the axis of
reals (the x-axis). If this limitation is agreed upon, then it will
be seen that the arguments of f are complex conjugates in pairs, and
likewise the sums of such pairs thus also will be complex conjugates.
Consequently, the sum total of the arguments is just twice the value
of the real part of each of the individual ones, and it follows that

$$\left(\frac{dw}{dz}\right)_x = 2\bar{A}\underline{R} \sum_{n=1}^{\infty} r_p^{2n-1} f\left[x + i(2n - 1)h\right] + 2A\underline{R} \sum_{n=1}^{\infty} r_p^{2n} f(x + i2nh)$$

$$= 2\underline{R}(A)\underline{R} \sum_{n=1}^{\infty} \left\{r_p^{2n-1} f\left[x + i(2n - 1)h\right] + r_p^{2n} f(x + i2nh)\right\} +$$

$$i2\underline{I}(A)\underline{R} \sum_{n=1}^{\infty} \left\{r_p^{2n} f(x + i2nh) - r_p^{2n-1} f\left[x + i(2n - 1)h\right]\right\} \qquad (2.4)$$

It may be noted that each term of the indicated sums is the sum
of two terms which involve the quantities $2n - 1$ and $2n$. It is con-
venient to reletter these multipliers as m and $m + 1$, respectively.
Hence one may now rewrite the expression for the complex velocity more
concisely as

$$\left(\frac{dw}{dz}\right)_x = 2\underline{R}(A)\underline{R}\sum_{m=1}^{\infty} r_p{}^m f(x + imh) + 2i\underline{I}(A)\underline{R}\sum_{m=1}^{\infty} \left(-r_p\right)^m f(x + imh) \quad (2.5)$$

Regardless of what the expression for the function f happens to
be, it is worthwhile pointing out that it is always possible to develop
equation (2.5) in a MacLaurin series provided one only moves away from
the origin a short distance, that is to say, provided the value of x
is small in comparison with the width of the tunnel, h. Upon carrying
out such a development one obtains the complex velocity as

$$\left(\frac{dw}{dz}\right)_x = 2\underline{R}(A)\underline{R}\sum_{m=1}^{\infty} r_p{}^m f(imh) + 2i\underline{I}(A)\underline{R}\sum_{m=1}^{\infty} \left(-r_p\right)^m f(imh) +$$

$$x\left\{2\underline{R}(A)\underline{R}\sum_{m=1}^{\infty} r_p{}^m\left[\frac{d}{dx}f(x + imh)\right]_{x=0} + \right.$$

$$\left. 2i\underline{I}(A)\underline{R}\sum_{m=1}^{\infty} \left(-r_p\right)^m\left[\frac{d}{dx}f(x + imh)\right]_{x=0}\right\} + \dots$$

In order to carry on the analysis any further it is necessary to
select a specific functional relationship for $f(x + imh)$ for each
particular case, depending upon what type of singularity is under study.

3.   In the case where the singularity is made up of a source of strength,
Q, and a vortex having a circulation, Γ, that is, for the case where
$A = Q + \frac{\Gamma}{i}$, the appropriate form for f is

$$f(x + imh) = \frac{1}{2\pi(x + imh)} \quad (3.1)$$

and consequently

$$\left(\frac{dw}{dz}\right)_x = \frac{Q}{\pi h} \, \underline{R} \sum_{m=1}^{\infty} \frac{r_p^m}{im} + \frac{\Gamma}{i\pi h} \, \underline{R} \sum_{m=1}^{\infty} \frac{(-r_p)^m}{im} +$$

$$\frac{x}{h} \left\{ \frac{Q}{\pi h} \, \underline{R} \sum_{m=1}^{\infty} \frac{r_p^m}{m^2} + \frac{\Gamma}{i\pi h} \, \underline{R} \sum_{m=1}^{\infty} \frac{(-r_p)^m}{m^2} \right\} + \ldots \qquad (3.2)$$

provided the series development is carried out only as far as the first term in x.

4.   In the case where the singularity is a doublet, of moment M, with its axis orientated so as to be parallel with the x-axis, the appropriate form for f is

$$f(x + imh) = + \frac{1}{2\pi(x + imh)^2} \qquad (4.1)$$

and consequently

$$\left(\frac{dw}{dz}\right)_x = -\frac{M}{\pi h^2} \, \underline{R} \sum_{m=1}^{\infty} \frac{r_p^m}{m^2} + \frac{2x}{h} \frac{M}{\pi h^2} \, \underline{R} \sum_{m=1}^{\infty} \frac{r_p^m}{im^3} + \ldots \qquad (4.2)$$

provided the series development is once again merely carried out as far as the linear term in x.

5.   In the preceding two Articles, one is confronted with the necessity of summing certain series which are expressed as follows:

$$\underline{R} \sum_{m=1}^{\infty} \frac{(\pm r_p)^m}{(im)^t} = \underline{R} \sum_{m=1}^{\infty} (\pm 1)^m \frac{\cos m\vartheta + i \sin m\vartheta}{(im)^t} \qquad (5.1)$$

where t = 1, 2, 3, $\cdots$.

   The latter form for the series results from acknowledgment of the fact that

$$r_p = \frac{2q - 3i - 1}{2q - i + 1} = \cos \vartheta + i \sin \vartheta \qquad (5.2)$$

inasmuch as $|r_p| = 1$. While it is also true that the restriction

$$-\pi \leq \vartheta = \arg \left(r_p\right) \leq 0 \qquad (5.3)$$

holds.

The case $\vartheta = 0$ corresponds to that of a solid wall boundary, while the case $\vartheta = -\pi$ corresponds to the situation where the boundary is a free fluid surface.

Now, depending on whether one employs the + sign or the - sign in equation (5.1), different trigonometric series are obtained, and likewise when different values of t are inserted into the formulas different trigonometric expressions are generated; but these are all easily summable, and the several cases of interest are presented below:

Case where $t = 1$ and the sign is +:

$$\underline{R} \sum_{m=1}^{\infty} \frac{r_p^{\ m}}{im} = \sum_{m=1}^{\infty} \frac{\sin m\vartheta}{m} = -\frac{\pi}{2} - \frac{\vartheta}{2} \qquad (5.4)$$

for ϑ lying within the open interval running from 0 to -2π, while

$$\underline{R} \sum_{m=1}^{\infty} \frac{r_p^{\ m}}{im} = 0 \quad \text{for} \quad \vartheta = 0 \quad \text{or} \quad \vartheta = -2\pi$$

Case where $t = 2$ and the sign is +:

$$\underline{R} \sum_{m=1}^{\infty} \frac{r_p^{\ m}}{m^2} \quad \sum_{m=1}^{\infty} \frac{\cos m\vartheta}{m^2} = \frac{\pi\vartheta}{2} + \frac{\vartheta^2}{4} + \frac{\pi^2}{6} \qquad (5.5)$$

for ϑ lying anywhere on the closed interval from 0 to -2π, including the end-points.

Case where $t = 3$ and the sign is +:

$$\underline{R} \sum_{m=1}^{\infty} \frac{r_p^{\ m}}{im^3} = \sum_{m=1}^{\infty} \frac{\sin m\vartheta}{m^3} = \frac{\pi\vartheta^2}{4} + \frac{\vartheta^3}{12} + \frac{\pi^2\vartheta}{6} \qquad (5.6)$$

for ϑ lying anywhere in the whole closed interval from 0 to -2π.

Case where $t = 1$ and the sign is -:

$$\underline{R} \sum_{m=1}^{\infty} \frac{\left(-r_p\right)^m}{im} = \sum_{m=1}^{\infty} (-1)^m \frac{\sin m\vartheta}{m} = -\frac{1}{2}\vartheta \qquad (5.7)$$

for ϑ lying within the open interval running from $-\pi$ to $+\pi$ while

$$\underline{R} \sum_{m=1}^{\infty} \frac{(-r_p)^m}{im} = 0 \quad \text{for} \quad \vartheta = \pm\pi$$

Case where $t = 2$ and the sign is $-$:

$$\underline{R} \sum_{m=1}^{\infty} \frac{(-r_p)^m}{m^2} = \sum_{m=1}^{\infty} (-1) \frac{\cos m\vartheta}{m^2} = \frac{\vartheta^2}{4} - \frac{\pi^2}{12} \qquad (5.8)$$

for ϑ lying anywhere on the closed interval from $-\pi$ to $+\pi$, including the end-points.

These expressions are all that is required in order to be able to describe the behavior of the velocities along the axis of symmetry of the wind tunnel in the neighborhood of the singularity. By substitution of the above-evaluated series in the expression given as equation (3.2), it is seen that the sought formula for the complex velocity along the axis of symmetry turns out to be, in the case of a singularity of the form $A = Q + \frac{\Gamma}{i}$:

$$\left(\frac{dw}{dz}\right)_x = -\frac{Q}{2\pi h} (\pi + \vartheta) - \frac{\Gamma}{2i\pi h} \vartheta +$$

$$\frac{x}{h} \left[\frac{Q}{2\pi h} \left(\frac{\pi^2}{3} + \pi\vartheta + \frac{\vartheta^2}{2} \right) + \frac{\Gamma}{2i\pi h} \left(\frac{\vartheta^2}{2} - \frac{\pi^2}{6} \right) \right] \qquad (5.9)$$

while if the properly evaluated trigonometric series are substituted into the expression given as equation (4.1) it is seen that the sought formula for the complex velocity along the axis of symmetry turns out to be, in the case of a doublet with moment M:

$$\left(\frac{dw}{dz}\right)_x = -\frac{M}{2\pi h^2} \left(\frac{\pi^2}{3} + \pi\vartheta + \frac{\vartheta^2}{2} \right) + \frac{2x}{h} \frac{M}{2\pi h^2} \left(\frac{\pi^2\vartheta}{3} + \frac{\pi\vartheta^2}{2} + \frac{\vartheta^3}{6} \right) \quad (5.10)$$

CHAPTER III

1.   After having gathered together all the necessary basic tools as derived above, it is possible now to go ahead with a study as to just how one might eliminate the interference effect at certain important locations in a perforated wind tunnel.

In this study two important limiting cases will be examined; viz., in one case the infinitesimally thin lifting wing will be treated (the antisymmetric problem), while in the other case the symmetric wing of finite thickness but at a zero angle of attack will be studied (the symmetric problem).

As is well known, the lifting wing (antisymmetric case) is handled by imagining that a vortex is concentrated at the quarter chord point of the airfoil with chordlength l, and the boundary condition that must be satisfied is that the streamlines become tangent to the mean camber line of the profile at the three-quarter chord point.

Now, in general, because of the interference effect of the surrounding wind tunnel, it will be true that the local angle of attack at the three-quarter chord point will no longer be the same as that angle given by the difference in direction between the $\vec{V}_\infty$ direction and the chord line of the profile, but instead this local angle of attack is increased by the amount

$$\epsilon = \frac{v_y}{V_\infty} \qquad\qquad (1.1)$$

where v_y is the component, taken in the direction of the y-axis, of the velocity induced at this point by interference action of the tunnel walls.

Consequently the boundary condition which now has to be satisfied is

$$\Gamma = \pi l (\alpha + \epsilon) V_\infty \qquad\qquad (1.2)$$

where Γ is the circulation existing around the profile in question, and, of course, α is the angle of attack of the chordline of the airfoil with respect to the free-stream direction, and l represents the chordlength.

If one wishes to eliminate the wall interference it is obvious that this means that one must counteract or cancel out the increment ϵ, and thus the v_y component must be annulled at the three-quarter chord point.

Let it now be assumed that the profile is located in the tunnel in such a way that the point where the vortex Γ is considered to be concentrated lies on the axis of the tunnel (the x-axis) and that this point is also the origin of coordinates. Thus the three-quarter chord point then lies at a distance $x = l/2$ downstream

By reference to equation (5.9) of Chapter II, it will be seen that under these stipulations one may write

$$v_y(l/2) = -\underline{I}\left(\frac{dw}{dz}\right)_{x=l/2} = -\frac{\Gamma}{2\pi h}\left[\vartheta - \frac{l}{2h}\left(\frac{\vartheta^2}{2} - \frac{\pi^2}{6}\right)\right] \qquad (1.3)$$

and thus the vertical component of induced velocity may be eliminated by making

$$\vartheta = \frac{2h}{l}\left[1 \pm \sqrt{1 + \frac{l^2}{h^2}\frac{\pi^2}{12}}\right] \qquad (1.4)$$

Since $\vartheta = \arg(r_p)$, it is less than zero and thus one must select the negative sign in the above formula.

On the other hand, if one would rather consider that ϑ is the fixed quantity, then it will be necessary to choose the l/h ratio so that

$$l/h = \frac{12\vartheta}{3\vartheta^2 - \pi^2} \qquad (1.5)$$

2. Now the symmetric problem is to be treated.

It is well known in this case that the symmetric profile can be simulated by employing a distribution of sources and sinks whose total strength is actually zero, and this distribution is equivalent (as far as its effects on the field of flow, at points far away from the location of these singularities, are concerned) to the effect of placing a doublet, whose moment is going to be proportional to the cross-sectional area of the profile, i.e., $M = -V_\infty S$, at the centroid of this area.

Let it be assumed, therefore, that the centroid of the profile is placed so as to coincide with the origin of the coordinate system for the tunnel flow, and let x be the abscissa value for the point at which one wants to cancel out any interference effect that would arise from action of the constraining wind tunnel walls. The magnitude of this induced velocity will thus be expressible as

$$\left(\frac{dw}{dz}\right)_x = -\frac{M}{2\pi h^2}\left[\left(\frac{\pi^2}{3} + \pi\vartheta + \frac{\vartheta^2}{2}\right) - \frac{2x}{h}\left(\frac{\pi^2\vartheta}{3} + \frac{\pi\vartheta^2}{2} + \frac{\vartheta^3}{6}\right)\right] \qquad (2.1)$$

and in order to cancel it, one must obviously impose the condition, there-
fore, that (since ϑ is to take on only negative values):

$$\frac{\pi^2}{3} + \pi\vartheta + \frac{\vartheta^2}{2} - \frac{2x}{h}\left(\frac{\pi^2\vartheta}{3} + \frac{\pi\vartheta^2}{2} + \frac{\vartheta^3}{6}\right) = 0 \qquad (2.2)$$

This relationship may be rearranged into a more convenient form to
work with, and upon carrying out this simplification it will be seen that

$$\frac{x}{h} = +\frac{3}{2}\frac{(\vartheta + 0.422\pi)(\vartheta + 1.578\pi)}{\vartheta(\vartheta + \pi)(\vartheta + 2\pi)} \qquad (2.3)$$

If the value of ϑ is confined to the closed interval running
between zero and $-\pi$, then the ratio x/h varies continuously from a
value of $+\infty$ to a value of $-\infty$ and passes through the point zero when
$\vartheta = -0.422\pi$.

The behavior of this relationship for x/h is of particular inter-
est in the neighborhood of the value where $\vartheta = -0.422\pi$ (or where
$x/h = 0$, that is). In the region of the ϑ close to $\vartheta = -0.422\pi$ the
expression for x/h may be developed in a series to give simply, to
first order, that

$$\frac{x}{h} = -0.462(\vartheta + 1.325) + \dots$$

where   (2.4)

$$\vartheta < 0$$

Conversely, it is easily seen what the expression for ϑ will be
as a function of x/h in this case, and this relationship may thus be
written as

$$\vartheta = -\frac{x}{h}(2.016) - 1.325 \qquad (2.5$$

In the case where an airfoil has central symmetry (both midchord
as well as fore-and-aft symmetry) it is obvious that the interference
from the wall will turn out to be zero at the location of the center of
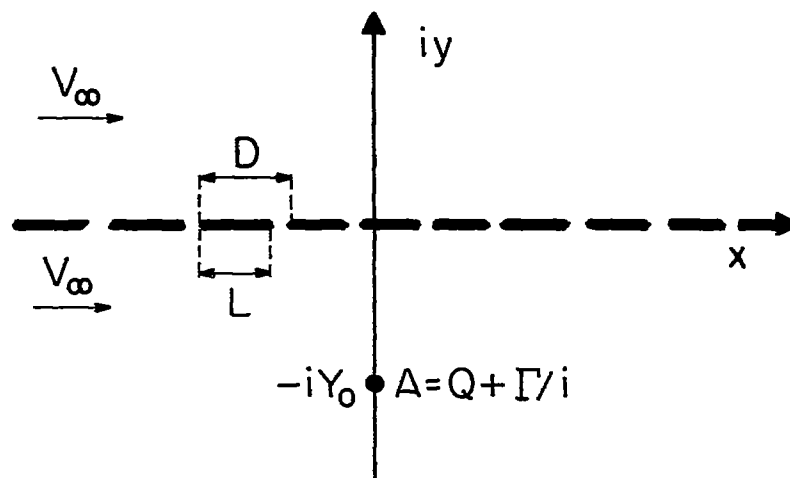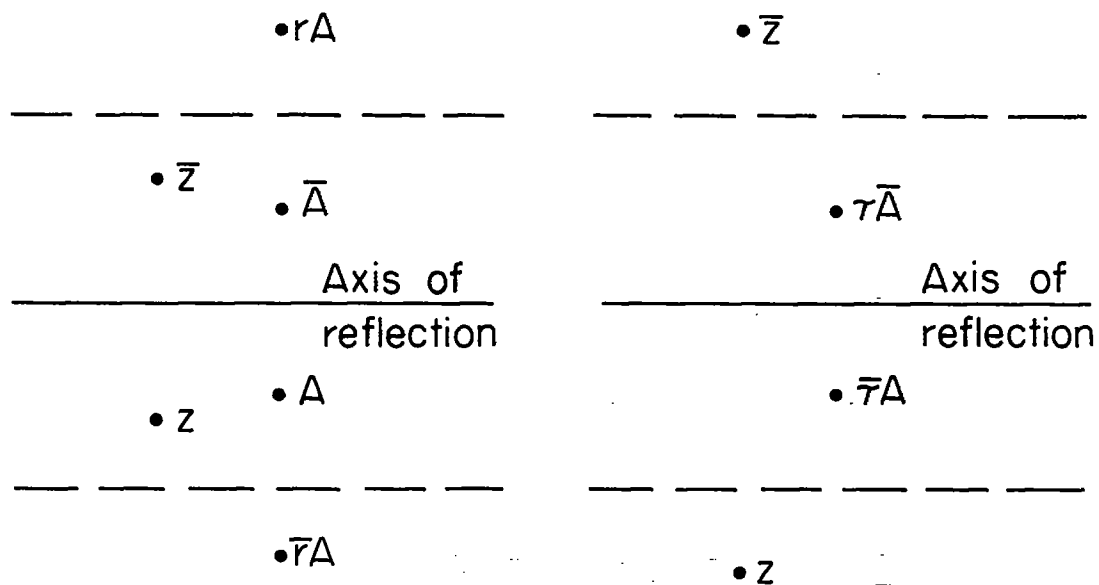symmetry, provided the value of the porosity is such that $\vartheta = -1.325$.

Figure 1

•r△

• $\overline{z}$

• $\overline{z}$ • $\overline{A}$

$\underline{\qquad\qquad}$ Axis of
reflection

• Z   • A

•$\overline{r}$△

(a) Field point inside.

• $\overline{z}$

• $\tau\overline{A}$

$\underline{\qquad\qquad}$ Axis of
reflection

• $\overline{\tau}$△

• Z

(b) Field point outside.

Figure 2

Figure 3

Figure 4